AMALGAMATED PRODUCTS OF C*-BUNDLES

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Abstract. We describe which classical amalgamated products of continuous C*-bundles are continuous C*-bundles and we analyse the involved extension problems for continuous C*-bundles.

Introduction

Different (fibrewise) amalgamated products of continuous C*-bundles have been studied over the last years ([1], [8], [6], [4]), one of the main questions being to know when these amalgamated products are still continuous C*-bundles.

In order to gather these different approaches in a joint survey, we first recall a few definitions from the theory of deformations of C*-algebras and we fix several notations which will be used in the sequel.

Then we present a few possible extension properties for continuous C*-bundles. More precisely, given a compact Hausdorff space X which is perfect, *i.e.* without any isolated point, we first recall in §2 that there is no general $C(X)$-linear version of the Hahn-Banach extension theorem for continuous $C(X)$-algebra. But we describe in §3 a Tietze extension property for continuous $C(X)$-algebras which will enable us to characterize in the following sections:
- when the canonical fiberwise amalgamated tensor products of a given continuous $C(X)$-algebra A with any other continuous $C(X)$-algebra B is a continuous $C(X)$-algebra ([6, Theorem 1.1 and Theorem 1.2]),
- when the canonical fiberwise amalgamated free products of a given continuous $C(X)$-algebra A with any other continuous $C(X)$-algebra B is a continuous $C(X)$-algebra ([4, Theorem 3.7 and Corollary 4.8]).

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1. $C(X)$-algebras

We recall first a few definitions from deformation theory for C*-algebras and we fix the notations which will be used in the sequel.

Let X be a compact Hausdorff space and $C(X)$ the C*-algebra of continuous functions on X with values in the complex field $\mathbb{C}$.

Definition 1.1. A $C(X)$-algebra is a C*-algebra A endowed with a unital $*$–homomorphism from $C(X)$ to the centre of the multiplier C*-algebra $\mathcal{M}(A)$ of A.

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Given a closed subset $Y \subset X$, we denote by $C_0(X \setminus Y)$ the closed ideal of continuous functions on X that vanish of Y. If A is a $C(X)$-algebra, then the subset $C_0(X \setminus Y).A$ is a *closed* ideal in A (by Cohen factorisation Theorem) and we denote by π_Y^X the quotient map $A \to A/C_0(X \setminus Y).A$.

If the closed subset Y is reduced to a point x and the element a belongs to the $C(X)$-algebra A, we usually write π_x, A_x and a_x for $\pi_{\{x\}}^X$, $\pi_{\{x\}}^X(A)$ and $\pi_{\{x\}}^X(a)$.

Note that the function

$$x \mapsto \|a_x\| = \inf\{\| [1 - f + f(x)]a\| \,;\, f \in C(X)\} \tag{1.1}$$

is always upper semi-continuous by construction. And the $C(X)$-algebra A is said to be **continuous** (or to be a *continuous C^*-bundle over* X) if the function $x \mapsto \|a_x\|$ is actually continuous for all a in A.

Definition 1.2. A *continuous field of states* on a unital $C(X)$-algebra A is a unital positive $C(X)$-linear map $\varphi : A \to C(X)$.

Remark 1.3. A (unital) separable $C(X)$-algebra A is continuous if and only if (iff) there exists a continuous field of states $\varphi : A \to C(X)$ such that for all $x \in X$, the induced state $\varphi_x : a_x \in A_x \mapsto \varphi(a)(x)$ is faithful on A_x ([2]).

2. Hahn-Banach extension properties

Given a compact Hausdorff space X, a continuous unital $C(X)$-algebra A, a unital $C(X)$-subalgebra $B \subset A$ and a continuous field of states $\phi : B \to C(X)$, there does not exist in general a $C(X)$-linear positive unital map $\varphi : A \to C(X)$ extending ϕ, *i.e.* a continuous field of states φ on A making the following diagram commutative:

$$\begin{tikzcd} B \arrow[r, "\phi"] \arrow[d, phantom, "\cap"] & C(X) \arrow[d, phantom, "\|"] \\ A \arrow[r, dotted, "\varphi"'] & C(X) \end{tikzcd}$$

The problem happens as soon as the interior of X is non empty. Indeed, consider:

- the compact space $X := \{0\} \cup \{\frac{1}{n}\,; n \in \mathbb{N}^*\}$,
- the unital continuous $C(X)$-algebra $A := C(X) \oplus C(X)$ and
- the $C(X)$-subalgebra $B := C(X).1_A + \Big(C_0(X \setminus \{0\}) \oplus C_0(X \setminus \{0\})\Big) \subset A$

And let $\phi : B \to C(X)$ be the continuous field of states on B fixed by the formulae

$$\phi(\,(b_1, b_2)\,)(\frac{1}{n}) = \begin{cases} b_1(\frac{1}{n}) & \text{if n is odd} \\ b_2(\frac{1}{n}) & \text{otherwise} \end{cases} \qquad \text{for} \quad (b_1, b_2) \in C_0(X \setminus \{0\}) \oplus C_0(Y \setminus \{0\})$$

Then, there cannot be any continuous field of states $\varphi : A \to C(X)$ such that $\varphi(b) = \phi(b)$ for all $b \in B$. Indeed, if $a = 1 \oplus 0 \in A$, one has that:

(1) $\varphi(a)(\frac{1}{n}) = 1$ if n is odd and
(2) $\varphi(a)(\frac{1}{n}) = 0$ otherwise.

Hence, the function $x \mapsto \varphi(a)(x)$ cannot be continuous at $x = 0$.

On the other hand, if Z is a second countable compact Hausdorff space Z and $X \subset Z$ is a non empty closed subspace, then any continuous field of states $\phi : \pi_X^Z(A) \to C(X)$ on the restriction $\pi_X^Z(A)$ can be extended to a continuous field of states $\varphi : A \to C(Z)$ by Michael continuous selection theorem (see e.g. [2, Proposition 3.13]), *i.e.* such that the following diagramme commutes:

$$\begin{tikzcd} \pi_X^Z(A) \arrow[r, "\phi"] & C(X) \\ A \arrow[u] \arrow[r, dotted, "\varphi"'] & C(Z) \arrow[u] \end{tikzcd}$$

3. Tietze extension properties

Given a second countable compact Hausdorff space X and a closed non empty subspace $Y \subset X$, we describe in this section when a continuous $C(Y)$-algebra A can be extended to X, *i.e.* when there exists a continuous $C(X)$-algebra D with a $C(Y)$-algebra isomorphic $\pi_Y^X(D) \cong A$.

If the C*-algebra A is an *exact* separable C*-algebra, then there exists a unital embedding of the $C(Y)$-algebra A into the trivial $C(Y)$-algebra $C(Y;\mathcal{O}_2) \cong C(Y)\otimes\mathcal{O}_2$, where $\mathcal{O}_2$ is the unital Cuntz C*-algebra generated by two isometries s_1, s_2 satisfying the relation $1_{\mathcal{O}_2} = s_1(s_1)^* + s_2(s_2)^*$ ([3]). Hence, the continuous $C(X)$-algebra $D := \{ f \in C(X, \mathcal{O}_2)\,;\, \pi_Y^X(f) \in A \}$ answers the question in that case.

But there are continuous $C(Y)$-algebras which are not exact C*-algebras. Thus, in order to study extensions in the general case, let us define in $X \times Y \times [0,1]$:

- the open subspace $U = \{(x,y,t) \in X \times Y \times [0,1]\,;\, 0 < t\}$ and
- the closed subspace $Z = \{(x,y,t) \in X \times Y \times [0,1]\,;\, 0 \leq t.d(x,Y) \leq 2d(x,Y) - d(x,y)\}$.

And let $\bar{d}$ be the metric on Z given by $\bar{d}((x,y,t),(x',y',t')) = d(x,x') + d(y,y') + |t-t'|$.

Proposition 3.1. *([6]) The coordinate map $p_1 : (x,y,t) \mapsto x$ gives a structure of $C(X)$-algebra on $C(Z)$ and the ideal $C_0(U \cap Z)$ is a continuous $C(X)$-algebra such that $C_0(U \cap Z)_{|Y} \cong C_0(Y \times (0,1])$, i.e. the map $(x,y,t) \in U \cap Z \mapsto x \in X$ is open.*

Proof. Given a function f in $C_0(U \cap Z)$, let us prove the continuity of the function

$$x \in X \mapsto \|\pi_x^X(f)\| = \sup\{|f(z)|\,;\, z \in p_1^{-1}(\{x\})\}$$

This map is already upper semi-continuous (u. s. c.) by construction. Hence, it only remains to show that for any point $x_0 \in X$ and any constant $\varepsilon > 0$, one has $\|\pi_x^X(f)\| > \|\pi_{x_0}^X(f)\| - \varepsilon$ for all points x in a neighbourhood of x_0 in X.

The uniform continuity of the function f implies that there exists $\delta > 0$ such that $|f(z) - f(z')| < \varepsilon$ for all z, z' in Z with $\bar{d}(z,z') < \delta$. Now three cases can appear:

1) If $x_0 \in Y$ and $x \in Y$ satisfies $d(x_0, x) < \delta/2$, then $|f(x, x, t) - f(x_0, x_0, t)| < \varepsilon$ for all $t \in [0, 1]$. And so $\|\pi_x^X(f)\| > \|\pi_{x_0}^X(f)\| - \varepsilon$.

2) If $x_0 \in Y$ and $x \in X \setminus Y$ satisfies $d(x_0, x) < \delta/4$, then for all $y \in Y$, the relation $d(x, y) \leq 2d(x, Y)$ implies that $d(y, x_0) \leq d(y, x) + d(x, x_0) \leq 2d(x, Y) + d(x, x_0) \leq \frac{3}{4}\delta$ and so $\left|f(x, y, t) - f(x_0, x_0, t)\right| < \varepsilon$ for all $t \in [0, 2 - \frac{d(x,y)}{d(x,Y)}]$. Whence the inequality $\|\pi_x^X(f)\| > \|\pi_{x_0}^X(f)\| - \varepsilon$.

3) If $x_0 \notin Y$ and the triple $(x_0, y_0, t_0) \in U \cap Z$ satisfies $|f(x_0, y_0, t_0)| = \|\pi_{x_0}^X(f)\| \neq 0$, then $d(x_0, y_0) < 2d(x_0, Y)$. Thus, there exists by continuity a constant $\alpha(x_0) \in]0, \delta/2[$ such that all $x \in X$ in the ball of radius $\alpha(x_0)$ around x_0 satisfy:

$$\text{a) } d(x, Y) > 0\,, \quad \text{b) } d(x, y_0) < 2d(x, Y)\,, \quad \text{c) } t_0 < 2 - \tfrac{d(x,y_0)}{d(x,Y)} + \delta/2\,.$$

And so $\|\pi_x^X(f)\| \geq \left|f(x, y_0, \inf\{t_0, 2 - \frac{d(x,y_0)}{d(x,Y)}\})\right| > \|\pi_{x_0}^X(f)\| - \varepsilon$. □

Remark 3.2. S. Wassermann pointed out that if $Y = \{0, 1\} \subset X = [0, 1]$, then $Z = \{(x, 0, t) \in [0, 1] \times \{0\} \times [0, 1]\,;\, t \leq \frac{2-3x}{1-x}\} \cup \{(x, 1, t) \in [0, 1] \times \{1\} \times [0, 1]\,;\, t \leq \frac{3x-1}{x}\}$. Hence, the $C(X)$-algebra $C(Z)$ is not continuous at $x = \frac{1}{3}$ and $x = \frac{2}{3}$.

But the essential ideal $C_0\big(Z \cap (0, 1] \times \{0, 1\} \times (0, 1]\big)$ is a continuous $C(X)$-algebra.

The following Corollary will be essential in the proof of Proposition 4.2.

Corollary 3.3. *Let X be a second countable compact space, $Y \subset X$ a non zero closed subset and A a continuous $C(Y)$-algebra.*

a) *$B := C(X) \otimes A \otimes C([0, 1])$ is a continuous $C(X \times Y \times [0, 1])$-algebra.*
b) *$D := [C_0(U).B]_{|Z} = C_0(U).B\big/C_0(U \backslash U \cap Z).B$ is a continuous $C(X)$-algebra.*
c) *There is an isomorphism of $C(Y)$-algebras $D_{|Y} \cong A \otimes C_0((0, 1])$.*

Proof. a) holds because the C*-algebras $C(X)$ and $C_0((0, 1])$ are nuclear.

b) Let $b \in D$. Then for all $x \in X$, we have

$$\|\pi_x^X(b)\| = \|b + C_0(X \setminus \{x\})D\| = \sup\{\|\pi_z^Z(b)\|\,;\, z \in p_1^{-1}(\{x\})\}\,,$$

whence the continuity of the map $x \mapsto \|\pi_x^X(b)\|$ by a) and Proposition 3.1.

c) One has $D_Y \cong [C_0(U).B]_Y \cong A \otimes C_0((0, 1])$ by Proposition 3.1. □

4. Amalgamated tensor products of continuous $C(X)$-algebras

Given a fixed compact Hausdorff space X, we study in this section the continuity properties of the different tensor products amalgamated over $C(X)$ of two given continuous $C(X)$-algebras A and B.

More precisely, let $A \odot B$ denote the algebraic tensor product (over $\mathbb{C}$) of A and B, let $\mathcal{I}_X(A, B)$ be the ideal in $A \odot B$ generated by the differences $af \otimes b - a \otimes fb$ ($a \in A$, $b \in B$, $f \in C(X)$) and let $A \underset{C(X)}{\odot} B$ denote the quotient of $A \odot B$ by $\mathcal{I}_X(A, B)$.

If $C_\Delta(X \times X) \subset C(X \times X)$ is the ideal of continuous function of $X \times X$ which are zero on the diagonal and $A \overset{m}{\otimes} B$ (resp. $A \overset{M}{\otimes} B$) is the minimal (resp. maximal) tensor product over $\mathbb{C}$ of the two *continuous* $C(X)$-algebras A and B, then the quotient $A \underset{C(X)}{\overset{m}{\otimes}} B := A \overset{m}{\otimes} B \big/ C_\Delta(X \times X) A \overset{m}{\otimes} B$ (resp. $A \underset{C(X)}{\overset{M}{\otimes}} B := A \overset{M}{\otimes} B \big/ C_\Delta(X \times X) A \overset{M}{\otimes} B$) is the minimal (resp. maximal) completion of the algebraic amalgamated tensor product $A \underset{C(X)}{\odot} B$. Further, the $*$-algebra $A \underset{C(X)}{\odot} B$ embeds in the $C(X)$-algebra $A \underset{C(X)}{\overset{m}{\otimes}} B$ ([1]) and we have

$$(4.1) \qquad \forall x \in X\,, \qquad (A \underset{C(X)}{\overset{m}{\otimes}} B)_x \cong A_x \overset{m}{\otimes} B_x \quad \text{and} \quad (A \underset{C(X)}{\overset{M}{\otimes}} B)_x \cong A_x \overset{M}{\otimes} B_x\,.$$

Let us also recall a characterisation of exactness given by Kirchberg and Wassermann.

Proposition 4.1. *(*[8, Theorem 4.5]*) Let $Y = \mathbb{N} \cup \{\infty\}$ be the one point compactification of $\mathbb{N}$ and let D be a C*-algebra. Then the following assertions are equivalent.*

i) *The C*-algebra A is exact.*
ii) *For all continuous $C(Y)$-algebra B, the minimal tensor product $A \overset{m}{\otimes} B$ is a continuous $C(Y)$-algebra with fibres $A \overset{m}{\otimes} B_y$ ($y \in Y$).*

It induces the following results for fibrewise tensor products of continuous $C(X)$-algebras.

Proposition 4.2. *(*[6], [4]*) Let X be a second countable compact Hausdorff space and A a separable unital continuous $C(X)$-algebra.*

If the topological space X is perfect (i.e. without isolated point), then the following assertions α_e) and β_e) (resp. α_n) and β_n)) are equivalent.

α_e) *The C*-algebra A is exact.*

β_e) *For all continuous $C(X)$-algebra B, the amalgamated tensor product $A \underset{C(X)}{\overset{m}{\otimes}} B$ is a continuous $C(X)$-algebra with fibres $A_x \overset{m}{\otimes} B_x$ ($x \in X$).*

α_n) *The C*-algebra A is nuclear.*

β_n) *For all continuous $C(X)$-algebra B, the amalgamated tensor product $A \underset{C(X)}{\overset{M}{\otimes}} B$ is a continuous $C(X)$-algebra with fibres $A_x \overset{M}{\otimes} B_x$ ($x \in X$).*

Proof. $\alpha_e) \Rightarrow \beta_e)$ If the C*-algebra A is exact, then the spatial tensor product $A \overset{m}{\otimes} D$ is a continuous $C(X \times X)$-algebra with fibres $A_x \overset{m}{\otimes} D_{x'}$ ($x, x' \in X$) ([8]). Hence, its restriction to the diagonal is as desired.

$\beta_e) \Rightarrow \alpha_e)$ Suppose conversely that the $C(X)$-algebra A satisfies β_e). And let us prove step by step that the C*-algebra A is exact.

Step a) All the fibres A_x are exact ($x \in X$). Indeed, given a point x in X, take a sequence of points x_n in X converging to x such that there is a topological isomorphism

$Y := \{x_n; n \in \mathbb{N}\} \cup \{x\} \cong \mathbb{N} \cup \{\infty\}$. Then, for any separable continuous $C(Y)$-algebra B, there is a continuous $C(X)$-algebra $\mathcal{B}$ such that $\mathcal{B}_{|Y} = B \otimes C_0((0,1])$ (Corollary 3.3). Now, the continuity of the $C(X)$-algebra $\mathcal{B} \overset{m}{\underset{C(X)}{\otimes}} A$ given by β_e) implies that of its restriction $\left(\mathcal{B} \overset{m}{\underset{C(X)}{\otimes}} A\right)_{|Y} \cong (C_0((0,1]) \otimes B) \overset{m}{\underset{C(Y)}{\otimes}} A_{|Y}$, whence that of the $C(Y)$-algebra $B \overset{m}{\underset{C(Y)}{\otimes}} A_{|Y}$ since there is an isometric $C(Y)$-linear embedding $B \hookrightarrow \mathcal{B}_{|Y}$. And this implies the exactness of the C*-algebra A_x by Proposition 4.1.

Step b) If B is a C*-algebra and $\mathcal{B}$ is the constant $C(X)$-algebra $C(X; B)$, then for all $x \in X$, we have the exact sequence

$$0 \to C_x(X) A \overset{m}{\otimes} B \to (A \overset{m}{\underset{C(X)}{\otimes}} \mathcal{B})_x = A \overset{m}{\otimes} B \to A_x \overset{m}{\otimes} B \to 0\,.$$

Step c) If B is a $C(X)$-algebra, then for all $x \in X$, we have the sequence of epimorphisms $(A \overset{m}{\underset{C(X)}{\otimes}} B)_x \twoheadrightarrow (A_x \overset{m}{\otimes} B)_x \twoheadrightarrow A_x \overset{m}{\otimes} B_x$

Step d) Now, let B be a C*-algebra, $K \triangleleft B$ a closed two sided ideal in B and take an element $d \in \ker\{A \overset{m}{\otimes} B \to A \overset{m}{\otimes} B/K\}$. Then for all $x \in X$, we have

$$\begin{aligned} d_x &\in \ker\{(A \overset{m}{\otimes} B)_x \to (A \overset{m}{\otimes} B/K)_x\} \\ &= \ker\{A_x \overset{m}{\otimes} B \to A_x \overset{m}{\otimes} B/K\} && \text{by } b) \\ &= A_x \overset{m}{\otimes} K && \text{by } a) \\ &= (A \overset{m}{\otimes} K)_x && \text{by } c) \end{aligned}$$

Thus, $d \in A \overset{m}{\otimes} K$. And so, the C*-algebra A is exact.

$\alpha_n) \Rightarrow \beta_n)$ has a similar proof to that of $\alpha_e) \Rightarrow \beta_e)$.

$\beta_n) \Rightarrow \alpha_n)$ If a C*-algebra A satisfies $\beta_n)$, then all the fibres A_x ($x \in X$) are nuclear by [8, Theorem 3.2] and so the C*-algebra A itself is nuclear (see e.g. [2, Proposition 3.23]). □

Remark 4.3. These characterisations do not hold anymore if the compact space X is not perfect. Indeed, if the space X is reduced to a point, then both the amalgamated tensor products $A \overset{m}{\underset{C(X)}{\otimes}} B$ and $A \overset{M}{\underset{C(X)}{\otimes}} B$ are constant, hence continuous.

Proposition 4.2 implies the following characterisation of exact continuous $C(X)$-algebras in the framework of $C(X)$-algebras.

Corollary 4.4. *Let X be a perfect compact metric space and A be a separable continuous $C(X)$-algebra. Then the following are equivalent*

(1) *The C*-algebra A is exact.*
(2) *For all exact sequence of continuous $C(X)$-algebras $0 \to J \to B \to D \to 0$, the sequence $0 \to A \overset{m}{\underset{C(X)}{\otimes}} J \to A \overset{m}{\underset{C(X)}{\otimes}} B \to A \overset{m}{\underset{C(X)}{\otimes}} D \to 0$ is exact.*

Proof. (2)⇒(1) If the unital continuous $C(X)$-algebra A satisfies (2) and the sequence $0 \to J_0 \to B_0 \to D_0 \to 0$ is an exact sequence of C*-algebras, then the sequence $0 \to J := C(X) \otimes J_0 \to B := C(X) \otimes B_0 \to D := C(X) \otimes D_0 \to 0$ is an exact sequence of $C(X)$-algebras. And the condition (2) implies the exactness of the sequence

$$0 \to A \overset{m}{\underset{C(X)}{\otimes}} J = A \overset{m}{\otimes} J_0 \to A \overset{m}{\underset{C(X)}{\otimes}} B = A \overset{m}{\otimes} B_0 \to A \overset{m}{\underset{C(X)}{\otimes}} D = A \overset{m}{\otimes} D_0 \to 0\,.$$

whence the exactness of A.

(1)⇒(2) If the $C(X)$-algebra A is an exact C*-algebra and $0 \to J \to B \to D \to 0$ is an exact sequence of $C(X)$-algebras, then the two first lines of the following diagram are exact by assumption (1)

$$\begin{array}{ccccc}
C_\Delta(X \times X) A \overset{m}{\otimes} J & \to & C_\Delta(X \times X) A \overset{m}{\otimes} B & \to & C_\Delta(X \times X) A \overset{m}{\otimes} D \\
\downarrow & & \downarrow & & \downarrow \\
A \overset{m}{\otimes} J & \to & A \overset{m}{\otimes} B & \to & A \overset{m}{\otimes} D \\
\downarrow & & \downarrow & & \downarrow \\
A \overset{m}{\underset{C(X)}{\otimes}} J & \dashrightarrow & A \overset{m}{\underset{C(X)}{\otimes}} B & \dashrightarrow & A \overset{m}{\underset{C(X)}{\otimes}} D
\end{array}$$

Besides, all the columns of the diagram are exact by definition, whence the exactness of the last line by a diagram chasing. □

5. Amalgamated free products of continuous $C(X)$-algebras

In this section, we describe the continuity properties of different free products amalgamated over $C(X)$ of two given unital continuous $C(X)$-algebras A and B.

Proposition 5.1. *(*[4]*) Let X be a second countable perfect compact Hausdorff space and A a separable unital continuous $C(X)$-algebra.*

Then the following assertions are equivalent.

α_e*) The C*-algebra A is exact.*

γ_e*) For all separable unital continuous $C(X)$-algebra B and all continuous fields of faithful states $\phi : A \to C(X)$, $\psi : B \to C(X)$, the reduced amalgamated free product $(C, \phi * \psi) = (A, \phi) \underset{C(X)}{*} (B, \psi)$ is a continuous $C(X)$-algebra with fibres $(C_x, \phi_x * \psi_x) = (A_x, \phi_x) * (B_x, \psi_x)$.*

Proof. $\gamma_e) \Rightarrow \alpha_e)$ Let B be a unital separable continuous $C(X)$-algebra and let ψ be a continuous field of faithful states ψ on B. Set $D = A \overset{m}{\underset{C(X)}{\otimes}} B$ and let E be the Hilbert D-bimodule $E = L^2(D, \phi \otimes \psi) \underset{C(X)}{\otimes} D$.

Then, the following assertions are equivalent ([4, Lemma 4.5]).

a) D is a continuous $C(X)$-algebra with fibres $D_x \cong A_x \overset{m}{\otimes} B_x$ $(x \in X)$.
b) The Pimsner C*-algebra $\mathcal{T}_D(E \oplus D)$ of the full Hilbert D-bimodule $E \oplus D$ is a continuous $C(X)$-algebra with fibres $\mathcal{T}_{D_x}(E_x \oplus D_x)$ $(x \in X)$.

But there is a $C(X)$-linear isomorphism $\mathcal{T}_D(E \oplus D) \cong C \rtimes \mathbb{N}$. And so, these two assertions are equivalent to the continuity of the reduced amalgamated free product $(C, \phi * \psi) = (A, \phi) \underset{C(X)}{*} (B, \psi)$ since the group $\mathbb{Z}$ is amenable ([2, Corollaire 5.10]).

$\alpha_e) \Rightarrow \gamma_e)$ Conversely, if A is an exact C*-algebra and B is a unital separable continuous $C(X)$, the amalgamated tensor product $D = A \overset{m}{\underset{C(X)}{\otimes}} B$ is a continuous $C(X)$-algebra with fibres $D_x \cong A_x \overset{m}{\otimes} B_x$ for $x \in X$ ([1]). Hence, the reduced amalgamated free product $(C, \phi * \psi) = (A, \phi) \underset{C(X)}{*} (B, \psi)$ is a continuous $C(X)$-algebra ([4, Theorem 4.1]).

□

Remark 5.2. There is no similar result for full amalgamated free product. Indeed, the full amalgamated free product $A \overset{f}{\underset{C(X)}{*}} B$ of two unital continuous $C(X)$-algebras A and B is always a continuous $C(X)$-algebra with fibres $A_x \overset{f}{*} B_x$ $(x \in X)$ ([4, Theorem 3.7]).

Sketch of proof. The algebraic amalgamated free product $A \underset{C(X)}{\circledast} B$ is a dense $C(X)$-submodule of the amalgamated Haagerup tensor product $A \overset{h}{\underset{C(X)}{\otimes}} B$, which itself is contained in the full amalgamated free product $A \overset{f}{\underset{C(X)}{*}} B$ ([9]). And for all $d \in A \underset{C(X)}{\odot} B$, one has

$$\|d_x\|_{A_x \overset{f}{*} B_x} = \|d_x\|_{A_x \overset{h}{\otimes} B_x} = \inf\left\{ \|\sum_i a_i a_i^*\|^{\frac{1}{2}}.\|\sum_i b_i^* b_i\|^{\frac{1}{2}} \;;\; d_x = \sum_i a_i \otimes b_i \right\}$$

$$= \sup\left\{ \left| \langle \xi, \sum_i \pi(a_i).\sigma(b_i)\eta \rangle \right| \;;\; \begin{array}{l} \xi, \eta \text{ unit vectors in the Hilbert space } \ell^2(\mathbb{N}) \\ \pi\,,\sigma \text{ unital } *-\text{rep. of } A_x,\, B_x \text{ on } \ell^2(\mathbb{N}) \end{array} \right\}$$

Hence, the map $x \mapsto \|d_x\|$ is both upper and lower semi-continuous if $d \in A \underset{C(X)}{\odot} B$.

The proof for the continuity of the map $x \mapsto \|d_x\|$ for elements d in the algebraic amalgamated free product $A \underset{C(X)}{\circledast} B$ is similar ([4]).

References

[1] E. Blanchard, *Tensor products of $C(X)$-algebras over $C(X)$*, Astérisque **232** (1995), 81–92.

[2] E. Blanchard, *Déformations de C*-algèbres de Hopf*, Bull. Soc. Math. France, **24** (1996), 141–215.

[3] E. Blanchard, *Subtriviality of continuous fieds of nuclear C^*-algebras*, J. Reine Angew. Math. **489** (1997), 133–149.

[4] E. Blanchard, *Amalgamated free products of C^*-bundles*, Proc. Edinburgh Math. Soc. **52** (2009), 23–36.

[5] E. Blanchard, E. Kirchberg, *Global Glimm halving for C^*-bundles*, J. Op. Th. **52** (2004), 385–420.

[6] E. Blanchard, S. Wassermann, *Exact C*-bundles*, Houston J. Math. **33** (2007), 1147–1159.

[7] S. Catterall, S. Wassermann *Continuous bundles of C^*-algebras with discontinuous tensor products*, Bull. London Math. Soc. **38** (2006), 647–656.

[8] E. Kirchberg, S. Wassermann, *Operations on continuous bundles of C*-algebras*. Math. Ann. **303** (1995), 677–697.

[9] G. Pisier, *Introduction to Operator space Theory.* London Math. Soc. Lecture Note Series **294**, Cambridge University Press (2003).

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